Representation Of Tensor Algebras In The Context Of Hilbert Module

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Abstract-In this Chapter, a Characterization is given of completely bounded representations of tensor algebra in terms of similarities of contractive intertwine. Also Proven is that for a C*-correspondence X over a C*-algebra A, $T(X) \otimes M_n$ and $T(X \otimes M_n)$ are isomorphic C*-algebra, where we denote the Toeplitz algebra of X and of $X \otimes M_n$ by T(X) and $T(X \otimes M_n$ respectively. The analogous statement for tensor algebra is deduced as a corollary.

Introduction-Let X be a C*-correspondence over a unital C*-algebra A with left action $\phi: A \longrightarrow L(X)$. We let $X^{\otimes n} = X \otimes_A X \otimes_A \ldots \otimes_A X$ be the n-fold internal tensor product of X as defined in (2.5) The left A-action is given by the *- homomorphism $\phi_n : A \longrightarrow L(X^{\otimes n})$ satisfying $\phi n(a)(x_1 \otimes x_2 \otimes \ldots \otimes x_n) = (\phi(a)x_1) \otimes x_{2\_\_\_} \otimes \ldots \otimes x_n$ By convention we declare $X^{\otimes 0} = A$ as a C*-correspondence over itself, as in (2.8), with the automorphism being the identity map $A \longrightarrow A$. Definition 3.1 The full Fock space F(X) over X is the C*-correspondence over A $\oplus^{\infty}_{n=0} X^n = A \oplus X \oplus (X \otimes_A X) \oplus \ldots$The left A-module structure is $\oplus_n \phi_n$ which we denote by ϕ_∞ It can be represented by the diagonal matrix

$$\phi_\infty(\alpha) = \begin{pmatrix} \alpha & & \\ & \phi(\alpha) & \\ & & \phi 2(\alpha) \end{pmatrix} \qquad \alpha \in A$$

Where $\phi_n(a)(\xi_1 \otimes \ldots \otimes \xi_1) = (\phi(a)\xi_1) \otimes \ldots \xi_n$ Looking at the (1,1) – entry , it is clear that ϕ_∞ is injective. Thus, we will often identify A with its image ϕ_∞ (A). For each $X \in X$, we define the creation operator $T_x \in L(F(X))$ by.

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$$T_x = \begin{pmatrix} 0 & & & \\ T_x^{(1)} & 0 & & \\ & T_x(2) & 0 & \\ & & T_x(3) & 0 \end{pmatrix}$$

Where $T_x(^k): X^{\otimes k} \longrightarrow X^{\otimes}(k+1)$ is given by the formula $T_x(^k)(X_1 \otimes \ldots \otimes X_K) = x_1 \otimes \ldots \otimes x_k$:

Definition 3.2 Let X be a C*-correspondence over A.

1. The tensor algebra of X, denoted T+(X), is the norm closed sub algebra of L (F(X)) generated by ϕ_∞ (A) and $\{T_x \mid x \in X\}$.
2. The Toeplitz algebra is the C*-algebra generated by $T_+(X)$ in L(F(x)).

Tensor algebras, introduced by Muhly and Soled in [.4], are non-self ad joint sub algebra of Toeplitz C*-algebras associated to X, which, in turn, were originally defined by Pilsner in [5]. Tensor algebras have an attractively tractable completely contractive representation theory expressed in terms of maps defined on the C*-correspondence X. The Purely algebraic tensor algebras have an analogous property, cf.[7].

Example 3.3 if A=X=C, then $F(C) \cong H^2$, the classical Hardy space and T_+ (X)= A(D) is the classical disc algebra.

Example 3.4 In case A = C AND $X = C^d$, then $T+(X) = A_d$ is Popescu's noncom-mutative disc algebra[6].

Example 3.5 Returning to Example 2.8 with $X = {}_aA$, we have $F(X) \cong \oplus_{\alpha n} A \cong 1^2(Z^+, A)$ Since A is unital with $1 \in A$, $T_+(X)$ is generated by ϕ (A) and a single creation operator $T_{1,}$ which we denote by S. furthermore, $T_+(x) \cong A \times \alpha^{Z+}$ The analytic (or non-self ad joint) crossed product of A by Z^+ determined by α , an operator algebra whose representations were studied in [8].

3.3 Completely Bounded Representations- Representations of T (X) and T_+ (X) are determined by representations of A and bimodal maps defined from X to B(H). We follow [31] in defining covariant representations of X.

Definition 3.6 A Pair (V,σ) is called a covariant representation of X on the Hilbert space H if .

1. $V:X \longrightarrow B(H)$ is linear,
2. $\sigma:A \longrightarrow B(H)$ is a non degenerate *-homomorphism, and

$V(xa) = V(x)\,\sigma(a)$ and $V(\phi(a)\,x) = \sigma(a)\,V(x) \;\forall\, x \in X,\ a \in A$:

We say (V,σ) is bounded (respectively completely bounded, contractive, completely contractive) when V is bounded (respectively, completely bounded contractive, completely contractive) when X is assigned the operator space structure it inherits form linking algebra

$$\mathfrak{I} = \begin{pmatrix} A & X^* \\ X & K(X) \end{pmatrix}$$

Since σ is a unital, completely contractive representation of a C*-algebra, then it is completely positive. Consequently, σ is a C*-representation. Using Muhly and Sole's characterization, π corresponds to a contraction $T:X \otimes_\sigma H \longrightarrow H$ satisfying then intertwining equation $T(\sigma^x\ 0\ \phi(\cdot))=(\cdot)T$: It is desirable to associate e T with a map $T:E \otimes_\sigma H \longrightarrow H$.However, balanced tensor products have only been defined when σ is a C*-homomorphism, here σ is only assumed to be a completely bounded homomorphism. This difficulty is easily dealt with by recognizing that the internal tensor product of C*-correspondences is a special case of the module Haagerup tensor product of operator modules, cf.[2] Thus, $E \otimes_\sigma H$ makes sense by considering H as an operator space with its column operator space structure $C_{H.}$ Then, CB(H)=B(H)and the following map makes sense.

$$I_x \otimes R : X \otimes_\sigma H \qquad X \otimes_\sigma H.$$

Observe that the property $R\sigma(a) = \sigma(a)\,R$ for all $a \in A$, is crucial. Denote by T the operator $R^{-1}\ T(I_x \otimes R): X \otimes_\sigma H \longrightarrow H$:Observe that $||T|| \le ||R^{-1}||\,||T||\,||I_x \otimes R|| \le ||R^{-1}||\,||R|| = ||\pi||_{cb}$ and $T(\phi(\alpha) \otimes I_H) = R^{-1}\,T^{\sim}(I_x \otimes R)\,(\phi(\alpha) \otimes I_H) = R^{-1}\,T^{\sim}\,(\phi((\alpha) \otimes I_H)\,(I_x \otimes R) = R^{-1}\,\sigma^{\sim}(\alpha)\,T^{\sim}\,(I_x \otimes R) = \sigma(\alpha)\,(R^{-1}\,T^{\sim}(I_x \otimes R) = \sigma(\alpha)\ T.$

References

1. D.P .Blecher. A new approach to Hilbert C-modules. Math. Ann., 307(2):253-290,1997.
2. D.P Blecher, P.S Muhly, and V.I Paulsen. Categories of operator modules (Morita equivalence and projective modules). Mem Amer. Math Soc., 143(681): Viii+94, 2000.
3. L.G Brown, P Green and M.A. Rieffel. Stable isomorphism and strong Morita equivalence of C*-algebras. Pacific J. Math., 71(2):349-363,1977.
4. P.S Muhly and B. Solel. Tensor algebras over C*-correspondences: representations, dilations, and C*-envelopes .J.Funct. Anal. 158(2):389-457,1998.
5. M.V. Pimsner. A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z. In free probability theory (Waterloo, ON , 1995), volume 12 of Fields Inst. Common., pages 189-212 . Amer. Math Soc. Providence, RI, 1997.
6. G. Popescu. Von Neumann inequality for (B(H)n)1. Math. Scand., 68(2) :292-304, 1991.
7. N.J. Fowler and I. Raeburn. The Toeplitz algebra of a Hilbert bi module. Indiana Univ. Math J., 48(1):155-181, 1999.
8. M.J. McAsey and P.S Muhly. Representations of nonselfadjoint crossed products. Proc. London Math. Soc. (3), 47(1); 128-144, 1983.