Ishikawa Iteration Process In L_p Space

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INTRODUCTION :

In this chapter we have discussed Ishikawa Iteration process in L_p apace which are of much importance in solving the problems related to economics and game theory. It is our purpose in this chapter to prove convergence theorems for both the Ishikawa iteration process and the Mann iteration process for continuous quasi-contractive mappings in L_p space for $1 < p \leq q$. Our method will, in addition, show that the comkpectness assumption on K as given by Chidume [9] and Rhoades [22], is not needed. For particular choices of the real sequences (a_n), (p_n) and (C_n) explicit convergence rates are calculated which, for p = 2, agree with results. The results of this chapter, together with earlier results of the authors chidume [9] and [6], then show that either the Ishikawa or Mann iteration process can be used to approximate the fixed point of a continous quasi-con-tractive mappings in L_p or I_p, 4 < p < . In all cases, no compactness assumption is needed. Furthermore, then main tools in our method of proof are of independent interest.

Let K be a subset of a Hilbert space H. A mapping $T : K \to R$ into itself is called quasi-contractive if there exists a constant $K \varepsilon [0,1]$ such that for each x, y e K,

$$\| Tx - Ty \| \leq K \max\{\| x - y \|, \| x - Tx \|, \| y - Ty \|, \| x - Ty \| \| y - T \times 1 \|\}$$
…..(1)

The contractive definition (1), apart from being an obvious generalization of the well know strict contraction mapping, is one of the most general opatractive definition for which picard iteration gives a unique fixed point. Several operators studied by various authors are actually quasi-contractive. Cric [10] proved that if (x, p) is a

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completemetric space and $T : x \to x$ is a continuous quasi-contractive mapping of X into itself then T has a unique fixed point in X. Rhoades [28] considered the following two fixed point iteration methods :

The Ishikawa iteration process by [13] is defined as follows : For K a convex subset of a Banach space X, and T a mapping of K into itself, the sequence $\{x_n\}_{n}^{\infty} = 0$ in K is defined by

x_0 e K

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T[\beta_n T x_n + (1 - \beta_n), x_n], n \geq 0$$

where $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ satisfy $0 \leq \alpha_n \leq \beta_n \leq 1$

for all n, $\lim\limits_{n \to \infty} \beta_n = 0$

$$\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \qquad \text{………[A]}$$

The man iteration process by Mann [16] and Rhoades [25] is defined as follows : with K, X and T as in [A], the sequence $\{x_n\}_{n=0}^{\infty}$ in K is defined by $x_0 \in K$

$$x_{n+1} = (1 - C_n) x_n + C_n \, T x_n, \, n \geq 0$$

where $\{C_n\}_{n=0}^{\infty}$ satisfies : $0 \leq C_n < 1$ for all n,

$$\lim_{n \to \infty} C_n = 0 \text{ and } \sum_{n=0}^{\infty} C_n = \infty \qquad \text{…….[B]}$$

In some application the condition

$$\sum_{n=0}^{\infty} C_n = \infty$$

is replaced by condition

$$\sum_{n=0}^{\infty} C_n (1 - C_n) = \infty$$

The iteration methods [A] and [B] have been studied by several authors and have been used to approximate fixed point of several nonlinear mapping, Although the iteration process [A] by using $\beta_n = 0$ for all n and putting different condition α_n examples given by Rhoades [25] show that the two iteration methods may exhibit different behaviours for different classes of nonlinear mappings. Rhoades [25] proved that most of the results established by Rhoades [23], using the Mann iternation process can be extended to the Ishikawa iteration scheme, hence providing a much larger clap of fixed point iternation procedures. Rhoades [25] also noted that the Mann iteration process can also be used to approximate fixed points of quasi-contractive mappings. He then posed the following question which remained open for many years : can the Mann iteration process be replaced by that of Ishikawa for continuous quasi-contractive mapping of K into itself, where K is compact convex subset of a Hilbert space. This question has recently been resolved in the affirmative by the author [9]. The author, in fact, proved that the Ishikawa iteration scheme converges strongly to fixed points of continuous quasi contractive mappings in L_p or l_p spaces $p \geq 2$, if $K^2 \leq (p-1)^{-1}$, where K is the constant appearing in inequality (1). The special case p = 2 then resolved the question raised by Rhoades. The author [6] proved that the Mann iteration process can also be used for continuous quasi – contractive mappings in L_p spaces for $2 \leq p \leq \infty$. The method used earlier [69], unfortunately could not be modified to provide any convergence theorem for either the Ishikawa process for the Mann process for continuous quasi-contractive mapping in L_p space when $1 \leq p \leq 2$.

Let X be a Banach space, we shall denote by J the normalized duality mapping from X to 2^{X^*} given by

$$Jx = \{f^* \varepsilon X^* : \| f^* \|^2 = \| x \|^2 = \leq x, f^* >\}.$$

where x* denotes the dual space of X and < > denotes the generalized duality pairing. If X* is strictly convex, then J is single – valued and if X* is uniformly convex, then J is uniformly continuous bounded sets.

THEORM (A) : Let j denote the single-valued normalized duality map on X and $X = L_p$, $1 \leq p \leq 2$. Then each x, y in X, the following inquality holds :

$$(p-1)\| x+y \|^2 \leq \| x \|^2 + \| y \|^2 + 2 \leq x, j(y) > \quad \ldots\ldots.(A_1)$$

REMARK 1 : Reich [3.] proved that if X* is uniformly convex then there exists a continuous non decreasing function.

$$a : [0, \infty) \to [0, \infty)$$

such that

$$a(0) = 0, a(ct) \leq ca(f) \text{ for all } c \geq 1$$

and

$$\| x+y \|^2 \leq x \|^2 + 2 \leq y, j(x) > + \max\{\| x \|, 1\} \| y \| b (\| y \|)\}$$

for all $x, y \ \varepsilon \ X$ ……..(2)

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