A Review Of The Fredholm Alternative & Compact Operators

Dr. Rakesh Kumar Singh*

ABSTRACT:

We review same features of the Fredholm alternative and compact operators. In more precise terms, the Fredholm alternative applies when K is a compact operator. From Fredholm theory, smoth integral kernels are compact operators and the refinements of the Fredholm alternative can be possible.

KEY WORDS :

Fredholmalternative, compact operator, Banach space & Hilbert space.

1. INTRODUCTION

In mathematics, the Fredholm alternative named after Ivar Fredholm is one of Fredholm's Theorems and is a result in Fredholm theory. It may be expressed in several ways, as a theorem of linear algebra, a theorem of operators. The applications of this theory play a vital rotle also in the theory of boundary value problems in differential equations.

The origin of the theory of compact operators is in the theory of integral equations whose integral operators supply concrete examples of such operators.

Let X be a n-dimensional vector space and $T \in L(X)$. We recall that the linear spaces $N(T)=\{x \in X : Tx=0\}$ & $B(T)=\{y \in X : Tx=y\}$ are called the null space of T & image space of T respectively. Setting $\alpha(T) = \dim N(T)$ & $\beta(T) = Co-\dim B(T)$. Then the following Fredholm alternative holds :

Either the equation Tx = 0 has only the trivial solution x = 0 in which case the equation Tx = y has a unique solution $x \in X$ for arbitrary $y \in X$ or

The equation Tx = 0 has non-trivial solutions is which case the equation Tx = y is not always solvable for each of $y \in X$

*__Department of Mathematics, Patna University, Patna__

The above alternative does not hold for infinite dimensional vector space but it is valid for certain class of integral equations.

Example :

Let us consider the vector space (s) of all sequences of real or complex numbers. The map defined on (s) by

$T(x_1, x_2......) = (0, x_1, x_2......)$ is one-one and onto, hence $T(x_1, x_2......) = 0$ has the only trivial solutions x1 = x2 = x3 = ...= 0, but the still the equation Tx = y is not solvable for all $y \in (s)$. It is solvable only for those $y \in (s)$. It is solvable only for those $y \in (s)$ whose first component is zero.

2. Fredhom has proved in his work (1903) that the alternative is valid for a certain class of linear integral equations.

$$x(s) - \int_a^b k(s,t)x(t)\,dt = y(s) \quad ....(1)$$

These equations are known as Fredholm integral equations of the 2nd kind. Riesz has constructed the theory of such equations on the property of the compact operator K.

Here K(s, t) is continuos on [a, b] × [a, b] and is called the Kernel of the integral equations, y(s) is also continous on [a, b] and therefore we have $|K(s,t)| < M$ for all $s,t \in [a,b]$ from which we get

$$|(Kx)(s)| \le \int_b^b |K(s,t)|\,|x(t)|\,dt$$

$$\le \mu\, |x|\, (b-a)$$

Therefore $\|Kx\| \le (b-a)\mu\|x\|$ and hence K is bounded .

We can write the integral equation (1) also in the following abstract form :

$x - Kx = (I-K)x = y$, where I is the identity operator on $C[a,b]$ and where the operator $K : C[a,b] \to C[a,b]$ is defined by

$$(Kx)\& = \int_a^b K(s,t)x(t)\,dt \quad (2)$$

Equations of this short are of great importance. A sufficient condition for the theorem to hold is for K (s, t) to be square integrable on the rectangle [a, b] × [a, b]

If in Fredholm alternative, we replace T bY (I - K) then the theorem can be restated as follows :

Either the equation (I - K) x = 0 has only the trivial solution x = 0 in which case (I - K) x = y has a unique solution $n \in C[a,b]$ for arbitrary $y \in C[a,b]$

or

The equation (I - K) x = 0 has non-trivial solution in $C[a,b]$ in which case

[I - K) x = y is not necessarly solvable through $x \in C[a,b]$ for arbitrary $y \in C[a,b]$

3. Results on the Fredholm operator and the compact operator generalise these results to vector spaces of infinite dimensions. Banach spaces and Hilbertspacs. The following results are the refinements of the Fredholm alternative.

Theorem (3.1)

If $K : E \rightarrow E$ is a compact operator on a normed linear space E, then the following alternative holds :

Either the Homogeneous equation possesses a non-trivial solution or the inhomogeneous equation

(i) $x - Kx = y$ (ii) $x' - K'x' = y'$

has for every $y \in E,\ y' \in E'$ unique solution $x \in E,\ x' \in E'$ (inparticular the solution x = 0 for y = 0 and $x' = 0$ *for* $y' = 0$)

Theorem (3.2)

If $K : V \rightarrow V$ is a compact operator on a Banach space v then either the operator (I - K) is invertible (i.e. bounded inverse) or there exists a non zero vector $x \in V$ such that Kx = x

Theorem (3.3)

If $K : V \rightarrow V$ is a compact operator on V then $\alpha(I-K) = \beta(I-K) < \infty$

Theorem (3.4)

If $K : H \rightarrow H$ is a linear, compact & self adjoint operator on a separable Hilber of space H and if $h \in H,\ \lambda \neq 0 \in R$ then the equation

$(\lambda I - K)g = h$ ....(3)

has a solution if and only if (g, H)=0 for all $g \in H$. The refined Fredholm alternative is also valid for $T = (\lambda I - K)$ on H.

For good proof of the above theorems spl see reference [1], [2], & [3]

REFERENCES

1. E.I. Fredholm : "Sur uneclasse d' equations functioneues" Act Math., 27 (1903)

2. A.G. Ramm : "A simple proof of the Fredholm alternative and a characterizations of the Fredholm operator"
American Mathematical monthly, 108 (2001), page 855.
3. Khvedelldze, B.V :Fredholm theorem for integral equations" in Haziwinkel, Michiel, Encyclopaedia of Mathematics, springer, 2001.
4. Rudin, W : "Functional analysis, International Series in pure & Applied Mathematics,
Mc Graw Hill, New York, 19 91
5. L.Nicolaescul : On the space of Fredholm operators, Math D.G./ 0005089, (2000)
6. H.G. Heuser : Functional Analysis, John Wiley & Sons, Chichester, U.K. 1982.

*****