A Studies of Ishikawa Iteration Process for Asymptotic Pointwise Mapping in Metric Spaces

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Abstract: Let (M, d) be a complete 2-uniformly convex metric space, C be a nonempty, bounded, closed and convex subset of M, and T be an asymptotic pointwise nonexpansive self mapping on C. In this paper, we define the modified Ishikawa
Iteration process in M and related theorem:

$$x_{n+1} = t_n T^n \left(S_n T^n (x_n) \oplus (1 - s_n)(x^n)\right) \oplus (1 - t_n) x_n$$

and we investigate when the Ishikawa iteration process converges.

Keywords: asymptotically nonexpansive mapping; fixed point; Ishikawa iteration process; uniformly convex metric space;

1. Introduction: The class of asymptotic non expansive mapping have been extensively studied in fixed point theory. Non expansive mapping in uniformly convex Banach spaces. The investigate the existence of a fixed point of asymptotic pointwise non expansive mappings and study the convergence of the modified Mann iteration in hyperbolic metric spaces.

2. Definition: Let (M, d) be a hyperbolic metric space. We say that M is uniformly convex if for any a ∈ M, for every r > 0, and for each ε > 0.

$$\delta(r, \varepsilon) = \inf\{1 - \frac{1}{r} d \left(\frac{1}{2} x \oplus \frac{1}{2} y, a\right) ; d(x, a).$$

Throughout this work, M is a hyperbolic metric space.

3. ISHIKAWA ITERATION PROCESS IN L_P SPACE

In this chapter we have discussed Ishikawa Iteration process in Lp space which are of such importance in solving the problems related to economics and game theory. It is our purpose in this chapter to prove convergence theorems for both the Ishikawa iteration process and the Mann iteration process for continuous quasi – contractive mappings in Lp space for $1 < p \leq q$. Our method will, in addition, show that the compactness

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Let K be a subset of a Hilbert space H. A mapping $T: K \to K$ into itself is called quasi contractive if there exists a constant $K \in [0,1]$ such that for each $x, y \in K$.

$$||Tx - Ty|| \leq K \max\left\{||x - y||, ||x - Tx||, ||y - Ty||, ||x - Ty|| ||y - Tx||\right\}, \ldots (1)$$

The contractive definition (1) apart from being an obvious generalization of the well known strict contraction mapping, is one of the most general contractive definitions for which Picard iteration gives a unique fixed point. Several operators studied by various authors are actually quasi – contractive. Cric [4] proved that if (X, p) is a completemetric space and $T: X \to X$ is a continuous quasi – contractive mapping of X into itself then T has a unique fixed point in X.

Rhodes [5] considered the following two fixed point iteration methods:

The Ishikawa iteration process by [6] is defined as follows: For K a convex subset of Banach space X, and T a mapping of K into itself, the sequence $\{X_n\}_n^{\infty} = 0$ in K is defined by

$x_0 \in K$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T\left[\beta_n T x_n + (1-\beta_n)x_n\right], n \geq 0 \quad \text{where} \quad \{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$$

satisfy $0 \leq \alpha_n < \beta_n < 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$ and

$$\sum_{n=0}^{\infty} \alpha_n \beta_n = \infty \text{ ....[A]}$$

The Mann iteration process by Mann [7] and Rhodes [8] is defined as follows: with K, X and T as in [A], the sequence $\{x_n\}_{n=0}^{\infty}$ in K is defined by

$$x_0 \in K$$

$$x_{n+1} = (1-C_n)x_n + C_n T x_n, n \geq 0$$

where $\{C_n\}_{n=0}^{\infty}$ satisfies : $0 \leq C_n < 1$ for all n,

$\lim_{n\to\infty} C_n = 0$ and $\sum_{n=0}^{\infty} C_n = \infty$ ....[B]

In some application the condition

$$\sum_{n=0}^{\infty} C_n = \infty$$

is replaced by condition

$$\sum_{n=0}^{\infty} C_n(1-C_n) = \infty$$

The iteration method [A] and [B] have been studied by several authors and have been used to approximate fixed point of several nonlinear mappings. Although the iteration process [A] by using $\beta_n = 0$ for all n and putting different condition examples given by Rhoades [8] show that the two iteration methods may exhibit different behaviours for different classes of nonlinear mappings. Rhoades [8] proved that most of the results established by Rhoades [9], using the Mann iteration process can be extended to the Ishikawa iteration scheme, hence providing a much larger clap of fixed point iteration procedures. Rhoades [8] also noted that the Mann iteration procedures. Rhoades [8] also noted that the Mann iteration process can also be used to approximate fixed points of quasi – contractive mappings.

Related Theorem:

Let X be a Banach space, we shall denote by J the normalized duality mapping from X to 2^{X^*} given by

$$Jx = \left\{f^* \in X^* : \left\|f^*\right\|^2 = \left\|x\right\|^2 = < x, f^* >\right\}$$

where x^* denotes the dual space of X and $< >$ denotes the generalized duality pairing. If X^* is strictly convex, then J is single – valued and if X^* is uniformly convex, then J is uniformly continuously bounded sets.

THEOREM (A): Let j denote the single – valued normalized duality map on X and X=Lp, 1: $p \leq 2$. Then for each x,y in X, the following inequality holds:

$(p-1)\left\|x+y\right\|^2 \leq \left\|x\right\|^2 + \left\|y\right\|^2 + 2 < x, j(y) >$................($A_1$)

REMARK 1: Reich [10] proved that if X^* is uniformly convex then there exists a continuous non decreasing function

$$a: [0,\infty) \to [0,\infty)$$

such that

$a(0) = 0, a(ct) \leq ca(f) \; for \; all \; c \geq 1$

and

$$||x+y||^2 \leq ||x||^2 + 2 < y, j(x) > + \max\{||x||, 1\}||y||b(||y||)\}$$

for all $x, y \in X$..............................(2)

LEMMA 1: For any real number $\lambda > 0$, the inequality:

$|\lambda x + (1-\lambda)y - z||^2 \leq (1-\lambda)\left\|y-z\right\|^2 + \lambda\left\|x-z\right\|^2 - (p-1)\lambda\left\|x-y\right\|^2 + \max\{||y - z||, 1\}\lambda||x-y||b(\lambda||x-y||)$

holds for all $x, y, z\epsilon X$, where $X = Lp, 1 < p \leq 2$.

PROOF: By using inequality (2)

$\left\|x + (1-\lambda)y - z\right\|^2 = \left\|(y-z) + \lambda(x-y)\right\|^2 \leq \left\|y-z\right\|^2 + 2\lambda < x-y, j(y-z) > \max\{||y-z||, 1\}\lambda||x-y||b(\lambda||x-y||) = \left\|y-z\right\|^2 + 2\lambda < x-z, j(y-z) > -2\lambda\left\|y-z\right\|^2 + \max[||y-z||, 1], 1||x-y||b(\lambda||x-y||)$

.... (L1.1)

Now, by inequality (A_1)

$\left\|y-z\right\|^2 = \left\|(z-x) + (y-z)\right\|^2 < \left(\frac{1}{p-1}\right)\left[\left\|z-x\right\|^2 + \left\|y-z\right\|^2 + 2 < z-x, j(y-z) >\right]$

so that

$(p-1)\left\|y-x\right\|^2 \leq \left\|z-x\right\|^2 + \left\|y-z\right\|^2 - 2 < x-z, j(y-z) >$

or

$2 < x-z, j(y-z) > \leq \left\|z-x\right\|^2 + \left\|y-z\right\|^2 - (p-1)\left\|y-x\right\|^2$

.... (L1.2)

Substitution of (L1.2) in (L1.1) yields

$\left\|\lambda x + (1-\lambda)y - z\right\|^2 \leq \left\|y-z\right\|^2 + \lambda\left\|z-x\right\|^2 + \left\|y-z\right\|^2 - (p-1)\lambda\left\|y-x\right\|^2 - 2\lambda\left\|x-z\right\|^2 + \max\{||y-z||, 1\} \; ||x-y||b(\lambda||x-y||)$

Which simplifies to give

$\left\|\lambda x(1-\lambda)y - z\right\|^2 \leq (1-\lambda)\left\|y-z\right\|^2 + \lambda\left\|x-z\right\|^2 - (p-1)\lambda\left\|y-x\right\|^2 + \max\{||y - z||, 1\} \; \lambda||x-y||b(\lambda||x-y||)$

as required.

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