Study of Weak Convergence And Fixed Point Theorems for Point – Dependent Mappings in Banach Spaces

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Abstract: The purpose of this article is to study the fixed point and weak convergence problem for the new defined class of point – dependent mappings relative to a Bregman distance Df in a Banach Space. We at first extend the Aoyama–Iemoto–Kohsaka -Takahashi fixed point theorem for mapping in Hilbert spaces in 2010 to this much wider class of nonlinear mapping in Banach Spaces.

Keywords: Fixed point, Bregman distance, subdifferential, Banach space.

1. Introduction: Let C be a nonempty subset of a Hilbert space H. A mapping $T: C \to H$ is said to be

1.1 Nonexpansive if $||Tx - Ty|| \leq ||x - y||, \forall x, y \in C$

1.2 Nonspreading if $||Tx - Ty||^2 \leq ||x - y||^2 + 2 < x - Tx, y - Ty >$

1.3 Hybrid if $||Tx - Ty||^2 \leq ||x - y||^2 + < x - Tx, y - Ty >, \forall x, y \in C$

2. Aoyama - Iemoto - Kohsaka - Takahashi Fixed Point Theorem:

Let C be a nonempty closed convex subset of a Hilbert Space H, and Let $T: C \to C$ be a l – hybrid mapping. Then, the following are equivalent

(a) There exists $x \in C$ such that such that $\{T^n x\}_{n \in C}$ is bounded.

(b) T has a fixed point.

Motivated by the above works, we extend concept of λ – hybrid from Hilbert spaces to Banach spaces.

3. Definition: For a nonempty subset C of a Banach Space X, a Gateaux differentiable Convex function $f: X \to (-\alpha, \alpha)$ and a function $l: C \to \square$ a mapping $T: C \to X$ is said to be point dependent λ – hybrid relative to Df if

$Df(Tx, Ty) \leq Df(x, y) + \lambda(y) < x - Tx; f(y) - f(Ty) >, \forall x, y \in C;$ where Df is the Bregman Distance associated with f and $f(x)$ denotes the Gateaux derivative of f at x.

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4. Definition: Suppose X be a Banach space. X satisfies Opial's condition if for each $x_0 \in X$ and each sequence $\{x_n\}$ weakly converging to x_0, the inequality

$\lim_{n\to\infty} \inf ||x_n - x|| > \lim_{n\to\infty} \inf ||x_n - x_0||$

holds for all $x \neq x_0$.

An equivalent definition is obtained by replacing the above inequality by:

$\lim_{n\to\infty} \sup ||x_n - x|| > \lim_{n\to\infty} \sup ||x_n - x_0||$

for all $x \neq 0$.

A Banach space X is said to satisfy weak Opial's if the following holds:

If a sequence x_n is weakly convergent to $x_0 \in X$, then for every $x \in X$.

$\lim_{n\to\infty} \inf ||x_n - x|| \geq \lim_{n\to\infty} \inf ||x_n - x_0||$;

equivalently

$\lim_{n\to\infty} \sup ||x_n - x|| \geq \lim_{n\to\infty} \sup ||x_n - x_0||$

Every Hilbert space and $1^P (1 \leq P < \infty)$ spaces satisfy Opial's condition and Banach spaces with weakly continuous duality mappings satisfy weak Opial's condition.

5. Weak Convergence And Fixed Point Theorems

Using the result of Reich [1], we have proved a theorem about common fixed points of a finite family of non-expansive mappings by iteration, where the sequence of iterates converges weakly to common fixed point. Also we have discussed an application of the iteration scheme to obtain an approximate solution of a system of equations.

Let C be a closed convex subset of a Banach space E, and let $T: C \to C$ be non-expansive. Baillon has shown that if $E = L^P (1 < P < \infty)$, and T has a fixed point, then for each x in C, $\{T^n x\}$ converge weakly to a fixed point of T. Also the purpose of this chapter is to point out that his ideas also lead to the following results. Recall that a sequence $\{x_n\} \subset E$ is weakly almost convergent to $y \epsilon E$ if $\left(\sum_{i=0}^{n-1} x_{i+k}\right)/n \to y$ uniformly in k, and that an operator $A \subset E \times E$ is said to be m-accretive if $R(I + A) = E$ and

$|x_1 - x_2| \leq |(x_1 - x_2) + r(y_1 - y_2)|$ for all $y_1 \in Ax_1, i = 1,2$ and $r > 0$.

6 (a) Suppose X be a uniformly convex Banach Space and K is a non-empty bounded closed convex subset of X, having weakly continuous duality mapping. If a sequence $\{x_n\} \subset K$ converges weakly to a point x_0,then x_0 is the asymptotic centre of x_n in K. Bose [4] has also proved that in a uniformly convex Banach Space if $F: K \to K$ is asymptotically non – expansive, then the asymptotic cente of $\{F^n x\}$ in K for any $x \in K$ is a fixed point of F.

(b) Let the mapping $F: K \to K$ be an asymptotically non expansive. And let K be a non – empty bounded closed convex subset of a uniformly convex Banach Space X. Suppose x_0 is the asymptotic centre of the sequence $\{F^n x\}$ for some $x \in K$. If the weak limit u_0 of a subsequence $\{F^{n_1} x\}$ is a fixed point of F, then it must coincide with x_0 which is a fixed point of F.

7. Related Theorem

Theorem A: Suppose $T: C \to C$ is a non – expansive mapping with a fixed point. Let C be a closed convex subset of uniformly convex Banach space E with a Frechet differentiable norm. Then $\{T^n x\}$ is weakly almost convergent to a fixed point of T.

Proof:

Suppose, $a_n = a_n(t) = |tx_n + (1-t)f_1 - f_2|(0 \le t \le 1)$, the modulus of convexity of the space, $M = |x_1 - f_1|, y(r) = (M/2)\,\delta(4r/M)$,

$S_{n,m} = T_{n-m-1}, T_{n-m-2} \ldots\ldots T_n$, and

$b_{n,m} = \left|S_{n,m}(tx_n + (1-t)f_1) - (tx_{n+m} + (1-t)f_1)\right|$.

Note that $a_{n,m} \le b_{n,m} + a_n$. After some manipulation we see that

$\gamma(|T(cx + (1-c)y) - cTx - (1-c)Ty|) \le |x-y| - |Tx - Ty|$

For all $0 < c < 1, |x-y| \le M$, and non-expansive $T: C \to C$. Hence

$\gamma(b_{n,m}) \le |x_n - f_1| = |x_{n+m} = f_1| \le \varepsilon_n \to 0$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} \sup a_n \le$ and $\lim_{n\to\infty} a_n(t) = a(t)$ exists.

Theorem B: Suppose $T: C \to C$ be a non – expansive mapping with a fixed point, and $\{c_n\}$ a real sequence such that $0 \le c_n \le 1$ and $\sum_{n=1}^{\infty} c_n(1-c_n) = \infty$, where C be a closed convex subset of a uniformly convex Banach space E with a Frechet differentiable norm. If $x_1 \in C$ and $x_{n+1} = c_n T x_n + (1-c_n)x_n$, then

$\{x_n\}$ converges weakly to a fixed point of T.

Proof: Since $\sum_{n=1}^{\infty} c_n(1-c_n) = \infty$, $\{x_n - Tx_n\}$ converges strongly to zero. Therefore every weak sub sequential limit of $\{x_n\}$ is a fixed point of T by using Browder [2].

Let f_1 and f_2 be two such limits. By the proposition 1, with $T_n = c_n T_n + (1-c_n)I)$, $(f_1 - f_2, f_1) = f_1 - f_2, f_2$ so that $f_1 = f_2$.

Theorem C: Suppose E be a uniformly convex Banach space with a Frechet differential norm, $J_r(r > 0)$ the resolvent of an m-accretive operator $A \subset E \times E$ with $0 \in R(A)$ and $\{r_n\}$ a positive sequence. Suppose that either

(i) $\{r_n\}$ is bounded away from zero, or

(ii) the modulus of convexity of E satisfies $\delta(\varepsilon) \ge K\varepsilon^p$ for some $K > 0$ and $p \ge 2$, and $\sum_{n=1}^{\infty} r_n^p = \infty$.

If $x_1 \in E$ and $r_{n+1} = J_{y_n} x_n$ for $n \ge 1$, then $\{x_n\}$ converges weakly to a zero of A.

Proof: Let $\gamma_{n+1} = (x_n - x_{n+1})/r_n$. In both cases $y_n \to 0$, Since $|x_n - J_1 x_n| \le |y_n|$, every subsequential weak limit of $\{x_n\}$ is a zero of A. Again the result now follows from the proposition 1 with $T_n = Jr_n$.

Let X be a uniformly convex Banach space and K be a closed convex subset of X. Suppose $\{T_1 : i = 1,2,\ldots,k\}$ is a family of non – expansive self – mapping of K. Define the following mappings:

Let $U_0 = I$, the identity mappings and

$U_1 = (1-\alpha)I + \alpha T_1 U_0$

$U_2 = (1-\alpha)I + \alpha T_2 U_1$

....

....

$U_k = (1-\alpha)I + \alpha T_k U_{k-1}$

where $0 < \alpha < 1$.

Consider a sequence $\{x_n\}$ in K defined by $x_1 \in K$ and

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_k U_{k-1} x_n$, ...(i)

$n = 1,2,3, \ldots\ldots\ldots\ldots\ldots\ldots$, where $\{\alpha_n\}$ is a real sequence such that $0 \le \alpha_n \le 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. We observe that for $k = 1$, the sequence (i) becomes,

$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1 x_n$, ... (ii)

which converges weakly to a fixed point of T_1 by the following theorem 4. Clearly the sequence (i) is a generalization of (ii).

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